ELSEVIER

Journal of Geometry and Physics 34 (2000) 111–120

JOURNAL OF GEOMETRY AND PHYSICS

Elasticity and conformal tension via the Kaluza–Klein mechanism

Manuel Barros[a,1], José L. Cabrerizo[b,*], Manuel Fernández[b]

[a] *Departamento de Geometría y Topología, Faculted de Ciencias, Universidad de Granada 18071, Granada, Spain*

[b] *Departmento de Geometrá y Topologia, Facultad de Matemáticas, Universidad de Sevilla, Apdo. de Correos 1160, 41080, Sevilla, Spain*

Received 10 June 1999; received in revised form 30 September 1999

Abstract

The principle of symmetric criticality allows one to reduce the search of symmetric critical points for the conformal total tension functional in a Kaluza–Klein conformal universe, to the search of closed curves in its gravitatory component which are critical points for certain r-elastic energy functionals. The constancy of the mean curvature is preserved in this reduction of symmetry. In this framework we study the r-elasticity of fibres in a semi-Riemannian warped tube. We obtain a wide class of tubes which are foliated by r-elastic circles and give some applications including a characterization of the photon sphere in a Schwarzschild spacetime from the point of view of the r-elasticity.

MSC: 53C40; 53C50
PACS: 02.40.Ky; 04.65.+e; 11.10.Kk

Keywords: Willmore–Chen submanifold; Kaluza–Klein metric; Conformal structure; r-elastica; Warped product metric; Conformal total tension functional
Subj. Class: Differential geometry; General relativity

* Corresponding author.
E-mail addresses: mbarros@goliat.ugr.es (M. Barros), jaraiz@cica.es (J.L. Cabrerizo), mafernan@cica.es (M. Fernández)
[1] Partially supported by a DGICYT Grant no. PB97-0784.

PII: S0393-0440(99)00057-1

1. Introduction

Let $(P, \bar{g})$ be a semi-Riemannian manifold and denote by $I(S, P)$ the smooth manifold of immersions of an n-dimensional compact smooth manifold S in P. The Willmore–Chen functional $\mathcal{T} : I(S, P) \to \mathbb{R}$ is given by (see [9])

$$\mathcal{T}(\varphi) = \int_S (\langle H, H\rangle - \tau)^{n/2} \mathrm{d}v,$$

where H is the mean curvature vector field of φ (i.e. the tension field of φ), $\mathrm{d}v$ denotes the volume element of the induced metric $\varphi^*(\bar{g})$ and τ stands for the extrinsic scalar curvature of φ. This functional is invariant under conformal transformations of the ambient space and is also known as the *conformal total tension functional*. The associated variational problem is actually stated in $(P, [\bar{g}])$, where $[\bar{g}]$ denotes the conformal structure defined by $\bar{g}$. The nicest immersions of S in $(P, [\bar{g}])$ are those which support the least possible conformal total tension from the surrounding conformal structure and therefore they provide Willmore–Chen submanifolds, i.e. critical points of $\mathcal{T}$. In particular [5], if G is an r-dimensional compact group of isometries in $(P, \bar{g})$, then this variational problem for $n = r + 1$ can be reduced to one associated with a certain r-elastic energy functional acting on closed curves in the orbit space (in the next section we define this r-elastic energy functional whose critical points are called *r-elasticae*, in particular, when $r=1$ this notion coincides with the classical one of elastic curve). This reduction of variables works fine when P is a principal fibre G-bundle with a principal flat connection over a certain semi-Riemannian manifold (N, g) and $\bar{g}$ is obtained by the so called Kaluza–Klein inverse mechanism. In addition, we are interested in those G-invariant Willmore–Chen submanifolds which have constant mean curvature in $(P, \bar{g})$ because they are solutions to the isoperimetric problem area–volume, which is what we call these G-submanifolds. Consequently, to obtain G-submanifolds in a Kaluza–Klein conformal universe, $(P, [\bar{g}])$, we only need to get r-elastic circles in its gravitatory component. Here circle means a closed curve with constant curvature.

In this paper, we consider spaces as $(N, g_\varepsilon^f) = M \times_f \varepsilon\mathbb{S}^1$, for a given semi-Riemannian manifold (M, g) and a smooth positive warping function f on M. For any $p \in M$, the fibre $\gamma_p = \{p\} \times \mathbb{S}^1$ provides a circle and then we characterize those fibres which are r-elastica. In particular, we classify those (M, g, f) which provide a foliation by r-elastic circles in (N, g_ε^f). We give some examples of solutions to illustrate this and use it to obtain examples of Kaluza–Klein conformal universes which are foliated with leaves being G-submanifolds. We also give some general applications to the existence of G-submanifolds. Finally, we obtain other convenient applications including the following one:

In the Schwarzschild spacetime, every great circle of the sphere whose distance to the center of the star is $(3+(1/r))m$ *(m being the mass of the star), is an r-elastica in this spacetime. Consequently, we can talk about a sequence of r-elastic spheres in the Schwarzschild spacetime which converges towards the photon sphere.*

2. Circle warped product

Let (M, g) be a semi-Riemannian manifold and f a smooth positive function on M. We denote by ∇ the Levi-Civita connection of (M, g). On $N = M \times \mathbb{S}^1$, we consider the semi-Riemannian metric $g_f^\varepsilon = g + \varepsilon f^2 \mathrm{d}t^2$, with obvious meaning and $\varepsilon = \pm 1$. Then, (N, g_ε^f) is called the *warped product* with *warping function* f, *base* (M, g) and *fibre* $(\mathbb{S}^1, \mathrm{d}t^2)$. Notice that the index of g_f^ε coincides with the index of g if $\varepsilon = +1$, while it increases the index of g if $\varepsilon = -1$, by one. When g is understood then, we still use $M \times_f \varepsilon\mathbb{S}^1$ to denote (N, g_ε^f) (see [7,14] for details about this subject). For any $p \in M$, let $\gamma_p = \{p\} \times_f \varepsilon\mathbb{S}^1$ be the fibre through p. We assume it is arclength parametrized, so ε is the causal character of its unit speed vector field $T = \gamma_p'$. Let ∇^f be the Levi-Civita connection of (N, g_ε^f). Then

$$\nabla_T^f T = -\frac{\varepsilon}{f}\mathrm{grad}(f), \tag{2.1}$$

where grad(f) stands for the gradient of f in (M, g). This equation shows that γ_p has constant curvature, say $\kappa \geq 0$, in (N, g_ε^f) for any $p \in M$. In particular, γ_p is a geodesic of (N, g_ε^f) if and only if p is a critical point of f. Let Σ be the set of critical points of f. Then the principal normal vector field to the fibres, U, defines a unit speed flow on the open set $M - \Sigma$ in the direction of grad(f), namely $\mathrm{grad}(f) = -\varepsilon\delta\kappa f U$, where δ denotes the causal character of U. Moreover, from $\nabla_T^f U = (U(f)/f)T$, one sees that the torsion of each fibre in (N, g_ε^f) vanishes identically.

Let Ω be the manifold of regular closed curves in (N, g_ε^f). For any natural number r, we define an r-elastic energy functional $\mathcal{F}^r : \Omega \to \mathbb{R}$ by

$$\mathcal{F}^r(\gamma) = \int_\gamma \left(\kappa^2\right)^{(r+1)/2} \mathrm{d}s,$$

where κ denotes the curvature function of γ and we write the integrand in that form to point out that it is an even function of the curvature. The variational problems associated with those functionals have been considered in [4]. Critical points of $\mathcal{F}^r$ are called *r-elasticae* and the Euler–Lagrange equations characterizing these curves were computed there.

Remark 1. *Recall that one of the oldest topics in the calculus of variations is the study of the elastic rod, which according to Daniel Bernouilli's idealization, minimizes total squared curvature among curves of the same length and first order boundary data. Therefore, the classical term elastica refers to a curve in the Euclidean plane $\mathbb{E}^2$ or Euclidean space $\mathbb{E}^3$ which represents such a rod in equilibrium. While the elastica and its generalizations have long been (and continue to be) of interest in the context of elasticity theory, the elastica as a purely geometrical entity seems to have been largely ignored. In [8] was found that the natural generalization of elastica to space forms (where arclength is not constrained) constitutes an interesting example in the setting of the general theory of exterior differential systems. The term of free elastica is used in [11] to name these curves. In [12], the elastic* *curves in $\mathbb{E}^2$ are used to obtain cylindrical configurations of Nambu-Goto-Polyakov string theory in $\mathbb{E}^3$. This result was drastically extended and simplified by the first author in [2]. In this paper, we use the term r-elastica to name the critical points of the functional $\mathcal{F}^r$. Results of Section 4 connecting this topic with the Willmore–Chen variational problem could be regarded as an intent to relate the r-elasticity actions with brane actions, in order to obtain symmetric shapes of p-branes coming from r-elasticae.*

Now, we deal with the following:

Problem. What about the r-elasticity of fibres in (N, g_ε^f)?

This problem has been studied in [5,6] when (N, g_ε^f) is a surface of revolution in either the Lorentzian 3-space $\mathbb{L}^3$ or the Euclidean 3-space $\mathbb{E}^3$, i.e. M is a plane curve and f denotes the distance to the axis of revolution on the profile curve. In particular, the so-called *trumpet surfaces* are characterized there as the only surfaces of revolution, besides right cylinders, whose parallels are all r-elastica.

Now, we use the Euler–Lagrange equations associated with $\mathcal{F}^r$ [4], to deduce that a fibre γ_p is an r-elastica if and only if

$$\kappa^r \left((r+1)R^f(U,T)T + \varepsilon\delta r\kappa^2 U\right) = 0, \tag{2.2}$$

along γ_p, where R^f stands for the curvature operator of (N, g_ε^f). Notice that Eq. (2.2) implies that the sectional curvature of (N, g_ε^f) along the osculating plane of an r-elastic fibre is either nonpositive (if $\varepsilon = +1$) or nonnegative (if $\varepsilon = -1$). It should also be observed that every geodesic fibre is automatically an r-elastica for any r. The following result provides a characterization of circle warped products (N, g_ε^f) whose fibres are all r-elastica.

Proposition 1. *All the fibres of $(N, g_\varepsilon^f) = M \times_f \varepsilon\mathbb{S}^1$ are r-elastica if and only if either*

1. *$f = a$ is a constant, $M - \Sigma = \emptyset$ and (N, g_ε^f) is the semi-Riemannian product of (M, g) and a circle of radius a (in this case all the fibres are geodesic), or*
2. *The unitary field $U = \mathrm{grad}(f)/\|\mathrm{grad}(f)\|$ defines a unit speed geodesic flow on $M - \Sigma$. Furthermore, the evolution of f along the U-flow is given by*

$$(r+1)f\,U(U(f)) = rU(f)^2. \tag{2.3}$$

Proof. The curvature term appearing in Eq. (2.2) is computed to be

$$R^f(U,T)T = -\frac{\varepsilon}{f}\nabla_U^f(\mathrm{grad}(f)).$$

We put $\mathrm{grad}(f) = \delta U(f)U$ in the above formula and use the fact that the base (M, g) defines a foliation on (N, g_ε^f) with totally geodesic leaves isometric with (M, g) to obtain

$$R^f(U,T)T = -\frac{\varepsilon\delta}{f}\left(U(U(f))U + U(f)\nabla_U U\right).$$

Now, we combine this equation with (2.2) and use $\kappa^2 = U(f)^2/f^2$ to obtain $\nabla_U U = 0$ and along this unit speed geodesic flow, the function f evolves according to (2.3). □

Remark 2. *If (M, g) is Riemannian and $\varepsilon = -1$, then (N, g_ε^f) is a standard static space-time. Also, if (M, g) is a Lorentzian solution given in the above proposition and $\varepsilon = +1$, then it admits a geodesic and irrotational unit vector field U. In particular, if this is timelike, then it is (at least locally) a proper time synchronizable observer field.*

3. Some examples

In this section, we describe a list of examples which provide solutions to the problem of having all fibres being r-elastica. The simplest case is that where M is one-dimensional and it includes the surfaces of revolution in $\mathbb{E}^3$ and $\mathbb{L}^3$. Therefore, our preliminary example was given in [6] (see there for some pictures).

Preliminary example (The trumpet surfaces). Let $\mathbb{R}^3$ endowed with the metric $g_\mu = \mathrm{d}x^2 + \mathrm{d}y^2 + \mu\,\mathrm{d}z^2$, $\mu = \pm 1$. Then $(\mathbb{R}^3, g_\mu)$ is either the Euclidean 3-space, $\mathbb{E}^3$, if $\mu = 1$ or the Lorentzian 3-space $\mathbb{L}^3$ if $\mu = -1$. For a certain constant c, we define in the $\{x, z\}$-plane an arclength parametrized curve $\beta(s) = (x(s), z(s))$ and

$$x(s) = cs^{r+1}, \qquad z(s) = \int_0^s \sqrt{1 - \mu(r+1)^2 c^2 t^{2r}}\,\mathrm{d}t.$$

Then, we rotate β around the z-axis to obtain a surface of revolution $S_\beta = I \times_{x(s)} \mathbb{S}^1$, I being the domain of β, in $(\mathbb{R}^3, g_\mu)$ whose parallels are all spacelike, r-elastica in S_β.

Example 1 (A constant scalar curvature metric which is r-elastica foliated). Let (M, g) be the Poincaré half-plane identified with the region $v > 0$ in $\mathbb{R}^2$ and the metric $g = (1/v^2)(\mathrm{d}u^2 + \mathrm{d}v^2)$ with constant Gaussian curvature –1. This metric is a warped product one. In fact, we put $t = \ln v$ and then, $g = (1/\exp 2t)\mathrm{d}u^2 + \mathrm{d}t^2$. We consider the smooth function $f : M \to \mathbb{R}$ given by $f(u, t) = t^{r+1}$. The curves $u =$ constant define a geodesic flow in the direction of $\mathrm{grad}(f)$ in (M, g). In other words, $U = \partial_t$ generates a unit speed geodesic flow in (M, g). Now, the function f is positive in the open set $V = \{(u, v) |\, v > 1\}$ and its evolution throughout the U-flow is described by (2.3). Consequently, $V \times_f \varepsilon\mathbb{S}^1$ has all the fibres being r-elastica.

Metrics with constant scalar curvature in dimension 3 are the next in interest after those with constant curvature. The scalar curvature function of $V \times_f \varepsilon\mathbb{S}^1$ is, up to a positive constant, $-1 + ((r+1)/\exp 2t)(1-r)$. In particular, it is constant when $r = 1$. In this case, we obtain a metric with negative constant scalar curvature on the tube $V \times_f \varepsilon\mathbb{S}^1$ which is foliated by free r-elastic circles.

Example 2 (Tubes around a Robertson–Walker spacetime). Let (M, g) be a Robertson–Walker spacetime, i.e. $M = (0, a) \times_h S$, where a is some positive real number, $h : (0, a) \to \mathbb{R}$ is a positive smooth function, (S, g_0) is a three-dimensional Riemannian manifold with constant curvature and then $g = -\mathrm{d}t^2 + h^2 g_0$, with obvious meaning. The Robertson–Walker flow which is generated by $U = \partial_t$ is a geodesic one and it provides a relativistic model of the flow of a perfect fluid. We define $f : M \to \mathbb{R}$ by $f(t, p) = t^{r+1}$ and use Proposition 1 to conclude that the tube $M \times_f \varepsilon\mathbb{S}^1$ has all the fibres being r-elastica.

Example 3 (Tubes around a Schwarzschild universe). The Schwarzschild solution of the empty space Einstein equation models the gravitational field outside an isolated, static, symmetric star. Staticity and spherical symmetry conditions are both satisfied on the product manifold $M = \mathbb{R} \times (0, a) \times \mathbb{S}^2$, for $a > 0$ endowed with a warped product metric as follows:

$$g = -F^2(s)\,\mathrm{d}t^2 + \mathrm{d}s^2 + G^2(s)\,\mathrm{d}\sigma^2,$$

where $(\mathbb{S}^2, \mathrm{d}\sigma^2)$ is the unit round sphere. Now, (M, g) is a solution of the empty space Einstein equation provided

$$F = G' = \sqrt{1 - \frac{2m}{G}},$$

for some constant m. The function G, defined on $(2m, +\infty)$ is usually interpreted as the distance from the center of the star. We define $f : M \to \mathbb{R}$ by $f(t, s, p) = s^{r+1}$. The spacelike vector field $U = \partial_s$ generates a unit speed geodesic flow in the direction of $\mathrm{grad}(f)$ on M and f is described by Eq. (2.3) when it evolves along this flow. Therefore, all the fibres of the tube $M \times_f \varepsilon\mathbb{S}^1$ are r-elastica.

Example 4 (A wide framework of solutions). Let (M, g) be a Riemannian manifold endowed with a non-trivial closed conformal field X, i.e. there exists a smooth function ψ on M with $\nabla_z X = \psi.z$, for every vector z tangent to M. This kind of structure has been widely studied in the literature not only for itself but also related to other subjects (see [13] and references therein). For example it is known that the set Γ of points where X vanishes has at most two points and outside of Γ, (M, g) is locally a warped product with a one-dimensional base. The unitary vector field $U = X/\|X\|$, defined in $M - \Gamma$, generates a unit speed geodesic flow. Therefore, we can consider a positive smooth function with gradient in the U-direction and evolving, at least locally, according to Eq. (2.3) along the U-flow to obtain an ample family of tubes which are foliated by r-elastic circles.

Example 5 (An example which is not a warped product). All the above exhibited solutions are, at least locally, warped product semi-Riemannian metrics with a one-dimensional base, the base generating a geodesic flow. However, we can get other solutions. To make clear this claim, we take an arclength parametrized curve, $\gamma(s)$ in $\mathbb{E}^3$ with positive torsion everywhere. Then, we consider the ruled surface generated on $\gamma(s)$ by the binormal lines, $M = \{\gamma(s) + vB(s) \mid s \in I \text{ and } v > 0\}$. Let g be the metric on M induced by the Euclidean one on $\mathbb{E}^3$. Then (M, g) is not a local warped product, $g = \left(1 + v^2\tau^2(s)\right)\mathrm{d}s^2 + \mathrm{d}v^2$, where τ denotes the torsion function of γ. However, it still admits a geodesic flow, $U = \partial_v$. Therefore, if we choose $f : M \to \mathbb{R}$ defined by $f(s, v) = v^{r+1}$, then all the fibres of the tube $M \times_f \mathbb{S}^1$ are r-elastica.

4. Some applications in conformal Kaluza–Klein supergravity

Let $(P, [\bar{g}])$ be a conformal universe, i.e. $[\bar{g}]$ is the conformal structure associated with a semi-Riemannian metric $\bar{g}$ on P. Assume that G is a r-dimensional, compact Lie group which acts on P through isometries of $(P, \bar{g})$. The best worlds to live in this universe are those compact submanifolds which satisfy the following three properties:

1. The submanifolds are G-invariants. This means they have a natural degree of established G-symmetry.
2. The submanifolds support the least global tension possible from the surrounding conformal structure $[\bar{g}]$. In particular, they must be critical points of the total tension functional, i.e. Willmore–Chen submanifolds in $(P, [\bar{g}])$.
3. The submanifolds are solutions to the isoperimetric problem area–volume (minimum area for a fixed volume or maximum volume for a fixed area). In particular, they must have constant mean curvature in $(P, \bar{g})$.

From now on, we will use the term of *G-submanifold* to name those submanifolds in a conformal universe $(P, [\bar{g}])$, which are G-invariant, Willmore–Chen and they have a constant mean curvature in $(P, \bar{g})$.

Next, we describe a class of conformal universes which have great interest in supergravity. We start from a semi-Riemannian manifold (N, g), the gravitatory space. Let H be a closed normal subgroup of the fundamental group $\pi_1(N)$ and $\phi : \pi_1(N)/H \to G$ a monomorphism, where G is a r-dimensional, compact Lie group. Then, we can define a principal fibre G-bundle $\pi : P \to N$ which admits a principal flat connection, ω, whose holonomy sub-bundle is the regular covering of N associated with H. The way to construct this couple (P, ω) is well-known (see [10] for example). If $\mathrm{d}\sigma^2$ is a bi-invariant metric on G, we can define the following semi-Riemannian metric $\bar{g}$ on P:

$$\bar{g} = \pi^*(g) + \omega^*(\mathrm{d}\sigma^2),$$

and this is called a *Kaluza–Klein* metric (also known as a *bundle-like* metric). It should be noticed that $\bar{g}$ is the only semi-Riemannian metric on P with the following property: $\pi : (P, \bar{g}) \to (N, g)$ *is a semi-Riemannian submersion with totally geodesic fibres isometric with* $(G, \mathrm{d}\sigma^2)$ *and horizontal distribution defined by* ω. Furthermore, the natural action of G on P is carried out by isometries of $(P, \bar{g})$. Then, we get a class of conformal universes in which a unified Kaluza–Klein theory can be constructed. This unifies the gravity g with the gauge potential ω. Now the problem is to study G-submanifolds in this kind of conformal universes. The surprising fact is that this problem for $(r + 1)$-dimensional G-submanifolds is equivalent to that of r-elasticae with constant curvature in the gravitatory component (N, g). This equivalence was shown in [4,5] (see also [1,3] for other conformal universes). However, we wish to make some comments about this symmetry reduction method because it uses the principle of symmetric criticality (in a formulation due to Palais [15]) which has been used in many applications of the Calculus of Variations, in particular in Physics, without being particularly noticed.

Proposition 2 (Barros et al. [4,5]). *Let S be an $(r+1)-$dimensional, compact submanifold in P. Then, S is a G-submanifold in a Kaluza–Klein conformal universe $(P,[\bar{g}])$ if and only if $S=\pi^{-1}(\gamma)$ and γ is an r-elastica with constant curvature in its gravitatory component,* (N,g).

Proof. Let S be a compact, $(r+1)$-dimensional smooth manifold and put $I(S,P)$ to denote the space of immersions of S in P. The subspace of G-invariant immersions is denoted by $I_G(S,P)$. Notice that it is a smooth submanifold of $I(S,P)$ which is guaranteed by the compacity of G. This submanifold can be identified with the space of complete lifts of closed curves in (N,g). Namely, $I_G(S,P)=\pi^{-1}(\gamma)/\gamma$ is a closed curve in N. A direct computation shows that the mean curvature function of $\pi^{-1}(\gamma)$ in $(P,\bar{g})$ is obtained by lifting the curvature function of γ in (N,g) and the extrinsic scalar curvature of $\pi^{-1}(\gamma)$ in $(P,\bar{g})$ vanishes identically. Since the conformal total tension functional $\mathcal{T}:(S,P)\to\mathbb{R}$ is invariant under conformal changes in the ambient metric $\bar{g}$, it is obviously G-invariant. In this framework, we can use the principle of symmetric criticality: *the G -invariant critical points of the conformal total tension functional on $I(S,P)$ are the critical points of this functional but restricted to the submanifold $I_G(S,P)$*. Finally, we compute this restriction to obtain a constant multiple of the r-elastic energy functional $\mathcal{F}^r$ acting on the space of closed curves in (N,g). □

Propositions 1 and 2 can be combined to obtain examples of Kaluza–Klein conformal universes which are foliated with leaves being G-submanifolds. To illustrate this method, we give a couple of examples.

Example 6. Let V be the open region of the Poincaré half-plane which was defined in Example 1. We choose any compact Lie group G with dimension r and define the function f on V, like in that example, to obtain $(N,g_\varepsilon^f)=V\times_f\varepsilon\mathbb{S}^1$. Let $\tilde{N}$ be the universal covering of N which can be regarded as a principal fibre $\mathbb{Z}$-bundle on N. For any real number λ with $(\lambda/\pi)\notin\mathbb{Q}$ (the set of rational numbers), we define $\phi_\lambda:\mathbb{Z}\to\mathbb{S}^1$ by $\phi_\lambda(a)=\exp\mathrm{i}a\lambda$ which is a monomorphism between $(\mathbb{Z},+)$ and $\mathbb{S}^1\subset\mathbb{C}$ viewed as a multiplicative group. Since G admits closed geodesics, we can extend ϕ_λ to a monomorphism, also called ϕ_λ, from $\mathbb{Z}$ to G. This is used to define, via the extending transition functions method, a principal fibre G-bundle $\pi:P_\lambda\to N$ which admits a principal flat connection, ω, with holonomy sub-bundle $\tilde{N}(N,\mathbb{Z})$. Now, we define $\bar{g}=\pi^*(g_f^\varepsilon)+\omega^*(\mathrm{d}\sigma^2)$ and apply Propositions 1 and 2 to conclude that $(P_\lambda,[\bar{g}])$ is foliated by $(r+1)$-dimensional G-submanifolds.

In particular, if we choose $G=\mathbb{S}^1$, we get $(P_\lambda,[\bar{g}])$ doubly foliated by Willmore tori with leaves cutting along a closed geodesic of $(P_\lambda,\bar{g})$. In fact, let $p\in P_\lambda$ be a point with $x=(x_1,x_2)=\pi(p)\in V\times\mathbb{S}^1$. Let γ_1 be any closed free elastica in (V,g) [11]. Then, $\beta_1=(\gamma_1,x_2)$ is still a closed free elastica in $V\times_f\varepsilon\mathbb{S}^1$ because the base defines a totally geodesic foliation in $V\times_f\varepsilon\mathbb{S}^1$. Also $\gamma_2=\{x_1\}\times\mathbb{S}^1$ is a closed free elastica in $V\times_f\varepsilon\mathbb{S}^1$. Then, $\pi^{-1}(\gamma_1)$ and $\pi^{-1}(\gamma_2)$ are Willmore tori in $(P_\lambda,[\bar{g}])$ through p. Moreover, $\pi^{-1}(\gamma_1)\cap\pi^{-1}(\gamma_2)=\pi^{-1}(x)$ is a geodesic of $(P_\lambda,\bar{g})$.

Example 7. The above construction can be transferred to the tube around the Schwarzschild metric already considered in Example 3. Therefore, if (M,g) is the Schwarzschild metric and f is chosen as in that example, then $M \times_f \varepsilon\mathbb{S}^1$ is foliated by r-elasticae. Now, given any r-dimensional, compact Lie group, G, we define a monomorphism $\phi_\lambda : \mathbb{Z} \to G$ and use it to construct a principal fibre G-bundle, $\pi : P_\lambda \to M \times_f \varepsilon\mathbb{S}^1$, endowed with a principal flat connection, ω. The conformal structure associated with $\bar{g} = \pi^*(g + \varepsilon f^2 \mathrm{d}t^2) + \omega^*(\mathrm{d}\sigma^2)$ in P_λ is foliated by $(r+1)$-dimensional G-submanifolds.

The method we have exhibited gives interesting examples of Kaluza–Klein conformal structures which are foliated by G-submanifolds. However, it also works to show the existence of G-submanifolds in certain conformal structures. We have chosen a pair of examples to illustrate this application.

Example 8. In this example, we show the existence of an r-elastic circle in the hyperbolic plane for any natural number r, and it can be also applied to the hyperbolic n-space. Let p_0 be a point of the hyperbolic plane $\mathbb{H}^2$, and denote by g_0 the standard metric on $\mathbb{H}^2$. We use polar coordinates to see that

$$(\mathbb{H}^2, g_0) = (0, +\infty) \times_f \mathbb{S}^1,$$

where $f : (0, +\infty) \to \mathbb{R}$ is defined by $f(t) = \sinh^2 t$. We use Proposition 1 to see that the circle $\{t\} \times \mathbb{S}^1$ is an r-elastica in $(\mathbb{H}^2, g_0)$ if and only if $\tanh^2 t = 1 + (1/r)$. As a consequence, we have proved the existence of G-submanifolds in a wide class of conformal structures in supergravity which have the hyperbolic space as gravity space. This class is a two-parameter family with one parameter being the isomorphy class of the Lie group G and the second one a rational number.

Example 9. In the Schwarzschild universe we can compute the distance from the center of the star for which each geodesic of the corresponding sphere is an r-elastica in that Schwarzschild universe. To do it, we notice that $\left(G'\right)^2 = (G - 2m)G$ and so $G'' = m/G$. Now, we use Proposition 1 to obtain that the above mentioned distance is achieved on the sphere $G^{-1}\left((3 + (1/r))m\right) \times \mathbb{S}^2$ and it gives solutions for any natural number r. Therefore, the circular planetary orbit $\left(t(u), G^{-1}\left((3 + (1/r))m\right), \theta(u)\right)$, projects in a great circle $\theta(u)$ of $\mathbb{S}^2$ which is an r-elastica. In this context, one could talk about r-elastic planetary orbits and r-elastic spheres in a Schwarzschild space time. Notice that the family of r-elastic spheres converges towards the photon sphere when r goes to infinity. In this sense, one could view the photon sphere (which gives orbits which are null geodesics) as a limit position for r-elasticity in the infinity.

References

[1] M. Barros, Willmore tori in non-standard 3-spheres, Math. Proc. Cambridge Phil. Soc. 121 (1996) 321–324.
[2] M. Barros, Symmetric configurations of classical string with rigidity, Preprint, 1999.
[3] M. Barros, The conformal total tension variational problem in Kaluza–Klein supergravity, Nucl. Phys. B 535 (1998) 531–551.
[4] M. Barros, A. Ferrández, P. Lucas, M.A. Meroño, Willmore tori and Willmore–Chen submanifolds in pseudo-Riemannian space, J. Geom. Phys. 28 (1998) 45–66.
[5] M. Barros, A. Ferrández, P. Lucas, M.A. Meroño, A criterion for reduction of variables in the Willmore–Chen variational problem and its applications, Trans. AMS, in press.

[6] M. Barros, O.J. Garay, Free elastic parallels in a surface of revolution, Amer. Math. Monthly 103 (1996) 149–156.
[7] A.L. Besse, Einstein Manifolds, Springer, Berlin, 1987.
[8] R. Bryant, P. Griffiths, Reduction for constrained variational problems and $\int \kappa^2 \mathrm{d}s$, Amer. J. Math. 108 (1986) 525–570.
[9] B.Y. Chen, Some conformal invariants of submanifolds and their applications, Boll. Un. Mat. Ital. 10 (1974) 380–385.
[10] S. Kobayashi, K. Nomizu, Foundations of Differential Geometry, Wiley, New York, 1963 (vol. I), 1969 (vol. II).
[11] J. Langer, D.A. Singer, The total squared curvature of closed curves, J. Differential Geom. 20 (1984) 1–22.
[12] Q.H. Liu, S.G. Zhang, Cylindrical configurations of classical string with rigidity, J. Phys. A 32 (1999) 5493–5499.
[13] S. Montiel, Stable constant mean curvature hypersurfaces in some Riemannian manifolds, Comment. Math. Helv. 73 (1998) 584–602.
[14] B. O'Neill, Semi-Riemannian Geometry, Academic Press, New York, 1983.
[15] R.S. Palais, Critical point theory and the minimax principle, Proc. Symp. Pure Math. 15 (1970) 185–212.